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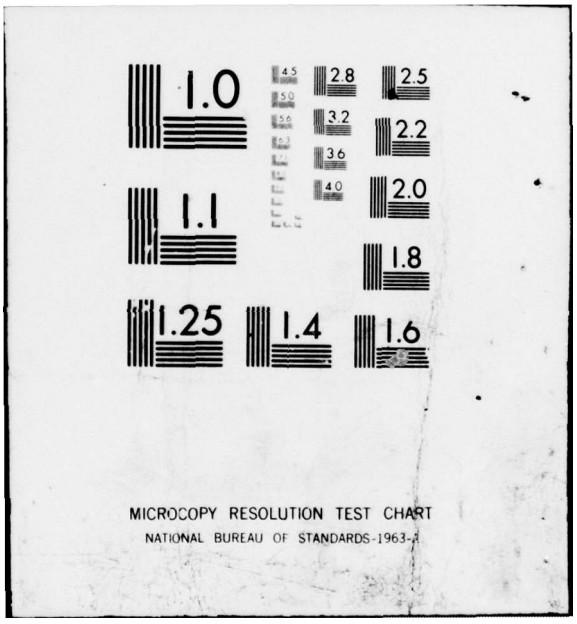
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A DRAZIN INVERSE FOR RECTANGULAR MATRICES

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In equation (3.7), page 8, the second = sign should be "minus".

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

(6) A DRAZIN INVERSE FOR RECTANGULAR MATRICES.

(10) R. E. Cline† and T. N. E. Greville

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ABSTRACT

The definition of the Drazin inverse of a square matrix with complex elements is extended to rectangular matrices by showing that for any B and W, m by n and n by m, respectively, there exists a unique matrix, X, such that $(BW)^k = (BW)^{k+1}XW$, for some positive integer k, XWBWX=X and BWX=XWB. Various expressions satisfied by B, W, X and related matrices are developed.

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SIGNIFICANCE AND EXPLANATION

Efficient methods for handling systems of linear simultaneous algebraic equations provide a fundamental tool in solving problems in almost every area of computing and applied mathematics. In matrix notation, the solution of $Ax = b$ is given by $x = A^{-1}b$, where A^{-1} denotes the inverse of A.

In classical matrix algebra, the inverse of A exists only if A is square and "nonsingular." The unique inverse then satisfies $AA^{-1} = A^{-1}A = I$. A singular or rectangular matrix has no inverse in this sense. However there may be associated with it a variety of "generalized inverses", each having some of the properties of the usual inverse. For example suppose that we have m equations in n unknowns, $Ax = b$, where A is $m \times n$, $m > n$, rank n. The solution in a least squares sense is

$$x = (A^TA)^{-1}A^Tb$$

and the matrix multiplying b is a generalized inverse.

A certain type of generalized inverse, the Drazin inverse, has heretofore been defined only for square (usually singular) matrices and has found application to Markov processes and to the solution of systems of ordinary differential equations. In this paper the definition of the Drazin inverse is extended to rectangular matrices and its properties studied. (This extended Drazin inverse is defined in the Abstract.)

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

A DRAZIN INVERSE FOR RECTANGULAR MATRICES

R. E. Cline and T. N. E. Greville

1. INTRODUCTION.

Let A be any rectangular matrix with complex elements. Then the Moore-Penrose inverse of A is the unique matrix $X = A^+$ such that

$$AXA = A,\ XAX = X,\ (AX)^H = AX,\ (XA)^H = XA \tag{1.1}$$

(where the superscript H denotes conjugate transpose). On the other hand, if A is square, the Drazin inverse of A is the unique matrix $X = A_d$ such that

$$A^k = A^{k+1}X, \text{ for some positive integer } k, \tag{1.2}$$

$$X = X^2A, \tag{1.3}$$

$$AX = XA, \tag{1.4}$$

It is the purpose of this paper to show that a Drazin inverse can be defined for rectangular matrices in such a way that both A^+ and A_d (when A is square) follow as special cases.

2. THE W-WEIGHTED DRAZIN INVERSE OF B.

Although the Drazin inverse was originally considered for elements in an associative ring [3] and Lemma 1 was established in that context [2], we use this result only for matrices and restate it accordingly.

LEMMA 1: For any matrices B and W, m by n and n by m, respectively,

$$(BW)_d = B(WB)_d^2W.$$

The reader can also verify Lemma 1 by taking X equal to the right member of the equation and $A = BW$ and verifying that (1.2)-(1.4) are satisfied.

Using (1.4) to rewrite (1.3) as $A_d = A_dAA_d$, the expressions in Corollary 1.1 now follow at once by induction.

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COROLLARY 1.1: For every positive integer p,

$$W(BW)_d^p = (WB)_d^p W$$

and

$$B(WB)_d^p = (BW)_d^p B.$$

Our first result, Theorem 2, is established for an arbitrary positive integer p. As will be indicated following the proof, however, the general case can always be reduced to $p = 2$ by a simple transformation.

THEOREM 2: For any matrices B and W, and for every positive integer p, there is a unique X such that

$$(BW)_d XW = (BW)_d^p , \tag{2.1}$$

$$BWX = XWB \tag{2.2}$$

$$BW(BW)_d X = X. \tag{2.3}$$

Also, there is a unique X such that

$$XW = BW(BW)_d^p , \tag{2.4}$$

$$WX = WB(WB)_d^p \tag{2.5}$$

$$XW(BW)^{p-1}X = X . \tag{2.6}$$

The unique matrix X which satisfies both sets of equations is

$$X = B(WB)_d^p . \tag{2.7}$$

Proof: Using (1.3) and Corollary 1.1, it is easily seen that X in (2.7) satisfies (2.1) to (2.6).

To establish uniqueness, we show first that (2.1), (2.2) and (2.3) imply (2.4), (2.5) and (2.6), and then that (2.4), (2.5) and (2.6) imply (2.7).

Now $XW = BW(BW)_d XW = BW(BW)_d^p$, by (2.3) and (2.1). Thus (2.4) holds, and combines with (2.3) to give

$$XW(BW)^{p-1}X = BW(BW)_d^p(BW)^{p-1}X = BW(BW)_d X = X .$$

Thus (2.6) also holds. Finally, using (2.2), (2.3), (2.4) and Corollary 1.1 we have

$$WX = WBW(BW)_d^X = (WB)_d WBWX$$

$$= (WB)_d WXWB = (WB)_d WBW(BW)_d^p B$$

$$= (WB)^2 (WB)_d^{p+1} = (WB)(WB)_d^p ,$$

that is, (2.5). Hence (2.1), (2.2) and (2.3) imply (2.4), (2.5) and (2.6).

If (2.4), (2.5) and (2.6) hold, then

$$X = XW(BW)^{p-1}X = BW(BW)_d^p (BW)^{p-1}X$$

$$= (BW)_d BWX = (BW)_d BWB(WB)_d^p$$

$$= BWB(WB)_d^{p+1} = B(WB)_d^p .$$

Hence (2.7) holds. ∎

Observe next that if $p \geq 1$, $q \geq -1$ and $r \geq 0$ are integers such that $q+2r+2 = p$, and if we let $(WB)^q = (WB)_d$ when $q = -1$, then $B(WB)_d^p = B(WB)^q[((WB)^r W)(B(WB)^q)]_d^2$. Consequently, considerations of X in Theorem 2 can always be reduced to the case $p = 2$ if B is replaced by $B(WB)^q$ and W is replaced by $(WB)^r W$.

COROLLARY 2.1: The matrix $X = B(WB)_d^2$ is the unique solution to the equations

(2.8) $(BW)^k = (BW)^{k+1}XW$, for some positive integer k ,

(2.9) $X = XWBWX$,

(2.10) $BWX = XWB$.

Proof: That $X = B(WB)_d^2$ is a solution is apparent by noting that the relations in (2.10) and (2.2) are identical, (2.9) is (2.6) when $p = 2$ and, with $XW = (BW)_d$ by (2.4), (2.8) is (1.2) for $A = BW$.

To show uniqueness, suppose both X_1 and X_2 are solutions to (2.8) for some positive integers k_1 and k_2, respectively, (2.9) and (2.10). Let $\hat{k} = \max(k_1, k_2)$. Then using repeated applications of equations (2.8), (2.9), and (2.10) we have

$$X_1 = X_1WBWX_1 = BWX_1WX_1 = (BW)^2(X_1W)^2X_1$$
$$= \ldots = (BW)^{\hat{k}}(X_1W)^{\hat{k}}X_1 = (BW)^{\hat{k}+1}X_2W(X_1W)^{\hat{k}}X_1$$
$$= X_2(WB)^{\hat{k}+1}W(X_1W)^{\hat{k}}X_1 = X_2WBW(BW)^{\hat{k}}(X_1W)^{\hat{k}}X_1$$
$$= X_2WBWX_1 .$$

Continuing in a similar manner, X_2WBWX_1 can be reduced to X_2. Thus $X = B(WB)^2_d$ is unique. ∎

It should be noted in Corollary 2.1 that $B_d = B(WB)^2_d$ when B is square and $W = I$. Also, there is a direct correspondence between the relations in (2.8) and (1.2), (2.9) and (1.3) when written as $X = XAX$, and (2.10) and (1.4) in which the role of W in (2.8), (2.9) and (2.10) is to act as a "sandwich" matrix so that products such as BWX and XWB can be defined. In view of the correspondence between the defining equations for A_d and those in Corollary 2.1, we define the Drazin inverse of a rectangular matrix in the following manner:

DEFINITION 1: For any matrices B and W, m by n and n by m, respectively, the matrix $X = B(WB)^2_d$ is called the <u>W-weighted Drazin inverse of B</u>, and is written as $X = B_{d,W}$.

Given a square matrix, A, the smallest positive integer k such that (1.2) holds is called the index of A, and it can be shown that k is the minimal power for which A^k and A^{k+1} have the same rank. Moreover, $(A_d)_d = A^2A_d$, so that $(A_d)_d = A$ if and only if A has index one, whereas $((A_d)_d)_d = A_d$, and A_d always has index one [3]. In Theorem 3 we characerize matrices, X, such that $X = B_{d,W}$ for some W in terms of matrices with index one.

THEOREM 3: Let B and X be any m by n matrices. Then $X = B_{d,W}$ for some W if and only if X has the form

(2.11) $$X = BYBYB$$

for some matrix Y such that both BY and YB have index one.

Proof: Suppose $W = Y(BY)^2_d = (YB)^2_d Y$, where BY and YB have index one. Then $WB = (YB)_d$ and, using (2.11),

$$WX = (YB)_d YBYB = YB = (WB)_d .$$

In a similar manner, $BW = (BY)_d$, and

$$XW = BY = (BW)_d .$$

Finally,

$$XWBWX = BYBYB = X ,$$

so that (2.4), (2.5) and (2.6) hold for B, W, X and $p = 2$. Thus $X = B_{d,W}$.

Conversely, if $X = B_{d,W}$ for some W, then

$$\begin{aligned} X = B(WB)^2_d &= (BW)_d B(WB)_d \\ &= BW(BW)^2_d B(WB)^2_d WB \\ &= B(WB)^2_d WB(WB)^2_d WB \end{aligned}$$

is of the form (2.11) where $Y = (WB)^2_d W$. Moreover, $BY = (BW)_d$ and $YB = (WB)_d$ have index one. ∎

For any matrix B in Theorem 3, both W and Y are n by m, and the following corollary shows that $W = Y$ and $X = B$ in (2.11) is necessary and sufficient to have B and $B_{d,W}$ equal.

COROLLARY 3.1: $B = B_{d,W}$ if and only if $B = BWBWB$.

Proof: If $B = BWBWB$, then $WB = (WB)^3$ implies $WB = (WB)_d$. Thus, $B = B_{d,W}$.

Conversely, if $B = B_{d,W} = B(WB)^2_d$, then $WB = (WB)_d$. ∎

3. THE B-WEIGHTED DRAZIN INVERSE OF W.

For any B and W, m by n and n by m, respectively, interchanging the roles of B and W throughout §2 provides a completely dual set of results for $W_{d,B} = W(BW)_d^2$, the B-weighted Drazin inverse of W. However, whereas $B_{d,W}$ has the size of B, $W_{d,B}$ has the size of B^H and B^+. Analogous to the observation in §2 that $B_d = B_{d,W}$ when B is square and $W = I$, we now consider choices of W for which $B^+ = W_{d,B}$.

Following the notation of [1], let $B^{(2)} = \{Z \mid ZBZ = Z\}$. Then $W_{d,B} \in B^{(2)}$ for any W such that $B^+ = W_{d,B}$.

LEMMA 4: $W_{d,B} \in B^{(2)}$ if and only if $W_{d,B} = W(BW)_d$.

Proof: If $W_{d,B} \in B^{(2)}$, then $W(BW)_d^3 = W(BW)_d^2$ so that $(BW)_d$ is idempotent.

Conversely, $W(BW)_d \in B^{(2)}$. ∎

An immediate consequence of Lemma 4 is that only expressions of the form $B^+ = W(BW)_d$ must be considered when determining choices of W. Moreover, observe that since $A^+ = A_d$ for any Hermitian matrix and $(BB^H)^+ = B^{H+}B^+$ for any B, then

$$B^+ = B^+BB^+ = B^HB^{H+}B^+ = B^H(BB^H)_d . \tag{3.1}$$

Therefore, one choice of W is B^H. To characterize the class of all matrices which can be used to form B^+ in this manner, it is convenient to replace W by C^H, where C has the size of B.

DEFINITION 2: For any m by n matrices B and C, C is said to be alias to B if $B^+ = C^H(BC^H)_d$.

Given any m by n matrix B with rank r, $B = EF$ is said to be a full rank factorization if E and F^H have r columns [1]. In this case $B^+ = F^+E^+$ with $E^+ = (E^HE)^{-1}E^H$ a left inverse of E and $F^+ = F^H(FF^H)^{-1}$ a right inverse of F.

Matrices alias to B are now characterized using a full rank factorization of B. In the proof of Theorem 5 we use the facts [4] that the general solution of a consistent system of equations $AXB = C$ can be written as

$$X = X_0 + Y - A^+AYBB^+ , \tag{3.2}$$

where X_0 is any particular solution and Y is arbitrary, and that necessary and sufficient conditions for the equations $AX = B$, $XC = D$ to have a common solution are that each is consistent and $AD = BC$ in which case

$$X_0 = A^+B + DC^+ - A^+ADC^+ \tag{3.3}$$

is a solution.

THEOREM 5: Let $B = EF$ be any full rank factorization. Then C is alias to B if and only if C has the form

$$C = ESF + Z \tag{3.4}$$

where S is any nonsingular matrix and Z is any matrix such that $BZ^H = 0$ and $Z^HB = 0$.

Proof; If C has the form in (3.4) where S and Z satisfy the hypotheses, then

$$\begin{aligned} C^H(BC^H)_d &= (C^HB)_dC^H = (F^HS^HE^HEF)_dF^HS^HE^H \\ &= F^HS^H(E^HEFF^HS^H)_dE^H \\ &= F^HS^H(S^H)^{-1}(FF^H)^{-1}(E^HE)^{-1}E^H \\ &= F^+E^+ = B^+ . \end{aligned}$$

Thus C is alias to B.

To prove the converse, suppose first that B has either full row rank or full column rank: If B has full row rank and C is alias to B, then $B^+ = B^H(BB^H)^{-1} = C^H(BC^H)_d$ implies that BC^H is nonsingular. Hence $C = S_1B$ where $S_1 = CB^H(BB^H)^{-1} = CB^+$ is nonsingular. By a similar type of argument, $C = BS_2$ with S_2 nonsingular if B has full column rank. Consequently, (3.4) holds with $Z = 0$ in both cases.

Suppose now that B has neither full row rank nor full column rank. With $B = EF$ a full rank factorization and C alias to B, then $F^+E^+ = C^H(BC^H)_d$ implies

$$E^+ = FC^H(EFC^H)_d$$

and

$$F^+ = C^H(EFC^H)_dE = C^HE(FC^HE)_d .$$

Therefore, CF^H is alias to E and E^HC is alias to F, so that there exist nonsingular matrices Q and R, say, for which

(3.5) $$FC^H = QE^H$$

and

(3.6) $$C^HE = F^HR.$$

Using (3.3) it then follows that

(3.7) $$X_0 = F^+QE^H + F^HRE^+ = F^+FF^HRE^+ = F^+QE^H$$

is a solution to (3.5), (3.6) and thus to

(3.8) $$FC^HE = QE^HE .$$

But by (3.2), all solutions of (3.8) can be written as

(3.9) $$C^H = X_0 + Z^H$$

where $Z^H = Y - F^+FYEE^+$ with Y arbitrary. Consequently those solutions in (3.9) such that (3.5) and (3.6) also hold must satisfy $FZ^H = 0$ and $Z^HE = 0$, which implies $BZ^H = 0$ and $Z^HB = 0$. Finally, the form in (3.4) is obtained from (3.9) by noting that the last expression for X_0 in (3.7) can be written as $X_0 = F^HS^HE^H$ with $S^H = (FF^H)^{-1}Q$ nonsingular. ∎

In Corollary 5.1 we characterize those matrices in (3.4) with $Z = 0$.

COROLLARY 5.1: For any full rank factorization $B = EF$, the set of matrices alias to B of the form $C = ESF$ is an equivalence class.

Proof: The relation is reflexive with $S = I$, that is, (3.1). Since $C = E(SF)$ is a full rank factorization, symmetry follows from $B = ES^{-1}(SF)$. Moreover, if D alias to C has the form $D = ET(SF)$ with T nonsingular, then D is alias to B. ∎

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